Combinatorial Duality and Intersection Product: A Direct Approach

Gottfried Barthel, Jean-Paul Brasselet,
Karl-Heinz Fieseler, Ludger Kaup

Abstract: The proof of the combinatorial Hard Lefschetz Theorem for the "virtual" intersection cohomology of a not necessarily rational polytopal fan that has been presented in [Ka] completely establishes Stanley's conjectures for the generalized h-vector of an arbitrary polytope. The main ingredients, namely, Poincaré Duality and the Hard Lefschetz Theorem, both rely on the intersection product. In the constructions of [BBFK$_2$] and [BreLu$_1$], there remained an apparent ambiguity. The recent solution of this problem in [BreLu$_2$] uses the formalism of derived categories. The present article gives a straightforward approach to combinatorial duality and a natural intersection product, completely within the framework of elementary sheaf theory and commutative algebra, thus avoiding derived categories.

Introduction

In [St], R. Stanley introduced the generalized h-vector for arbitrary polytopes. For rational polytopes, this new combinatorial invariant agrees with the vector of even (middle perversity) intersection cohomology Betti numbers of a projective toric variety associated with the polytope and then, they enjoy the same properties. Stanley proved that the Dehn-Sommerville equations (i.e., Poincaré duality) remain valid in the general case, and he conjectured that non-negativity and unimodality up to the middle dimension also should continue to hold. In the rational case, the unimodality property follows from the "Hard Lefschetz Theorem" for the intersection cohomology of a projective variety.

This conjecture motivated the search for a purely combinatorial approach to the intersection cohomology of toric varieties. Such an approach has been developed independently in [BBFK$_2$] and by P. Bressler and V. Lunts in [BreLu$_1$]. The basic idea in both articles is to view a (not necessarily rational) fan as a finite topological space, endowed with the topology given by the subfans as non-trivial open sets, and to study the properties of a certain sheaf on that "fan space" that agrees with the equivariant intersection cohomology sheaf for the associated toric variety in the rational case. This approach then yields a "virtual" intersection cohomology theory for the class of "quasi-convex" fans that includes all complete and hence, in particular, all polytopal fans. In [BBFK$_2$], the working principle was to present everything on a fairly elementary level, using geometry and commutative algebra only and avoiding the use of derived categories.

At the time when these articles were written, a purely combinatorial version of the Hard Lefschetz Theorem (HLT), as stated in the third section, was still lacking. This was the only missing piece to prove that the vector of even "virtual" intersection cohomology Betti numbers of a polytopal fan agrees with the generalized h-vector of the polytope, and thus, to fully establish Stanley's conjecture. As another problem, in the construction of the intersection product on the virtual equivariant intersection cohomology sheaf, apparently non-canonical choices entered.

In the meantime, a proof of the combinatorial Hard Lefschetz Theorem has been presented by K. Karu in [Ka]. The proof heavily relies on the study of the intersection product, since what actually is shown are the Hodge-Riemann bilinear relations (HRR) for the "primitive" (virtual) intersection cohomology, which imply HLT as an easy consequence. The apparent ambiguity in the definition of the intersection product, however, makes the argumentation quite involved, since one has to carefully keep track of the choices made. A first simplified version has recently been presented by Bressler and Lunts in [BreLu$_2$], using the framework of derived categories. In particular, they verify by a detailed analysis that none of the possible choices affects the definition of the pairing.

Our goal is to go one step further, namely, to give a short, direct, and elementary approach to duality and the intersection product in the "geometrical" spirit of [BBFK$_2$], following ideas of [Bri], the only prerequisites being sheaf theory and commutative algebra. While in [BreLu$_1$,2] the duality functor is apriori only defined as an endofunctor of a big derived category containing the "pure sheaves" as invariant subcategory, we construct the dual of a pure sheaf directly as a pure sheaf, avoiding the above detour. The crucial step is the definition of the restriction homomorphisms from a cone to a facet; the image of a section can be looked at as a kind of residue along the facet. The structure of the proof of Poincaré duality and the naturality of the intersection product is the classical one, as in [BreLu$_2$], with the single steps easily accessible. For the convenience of the reader we give here a complete presentation, referring always to the corresponding statements in [BreLu$_1$,2]. An intersection product corresponds to a sheaf homomorphism $\vartheta\colon \mathcal{E} \to \mathcal{D}\mathcal{E}$ of degree zero from the (graded) equivariant intersection cohomology sheaf $\mathcal{E}$ to its dual, and we check that $\mathcal{E}$ is self-dual in a natural way.

Our main result, cf also [BreLu$_2$] is stated below in such a way that it fits into the inductive proof of the Hard Lefschetz Theorem as given in [Ka]: Assuming HLT for polytopal fans in dimension $d < n$, the "Poincaré Duality Theorem" yields a natural intersection product on every fan in dimension n. In [Ka] it is shown that HRR for simplicial fans in any dimension – which is valid by [Mc] – together with HRR for arbitrary fans in dimensions $d < n$ imply HRR and thus, HLT in dimension n. In that induction step, it is most useful to work with a canonical pairing.

To state our result, we use this notation, explained more systematically in section 0: Let Δ be a quasi-convex fan in a vector space V of real dimension n with a fixed volume form, and $\partial\Delta$ its boundary fan. The global sections of the equivariant intersection cohomology sheaf $\mathcal{E}$ on Δ and on $(\Delta, \partial\Delta)$, respectively, are (graded) modules over the (graded) symmetric algebra $A := S(V^*)$ of polynomial functions on V.

Poincaré Duality Theorem. [BreLu$_2$, 3.16] *In the above situation, let us assume that the Hard Lefschetz Theorem holds in all dimensions below n. Then there is a natural intersection product*

$$\text{(PD)} \qquad \mathcal{E}(\Delta) \times \mathcal{E}(\Delta, \partial\Delta) \quad \longrightarrow \quad A[-2n]$$

giving rise to a dual pairing of finitely generated free A-modules.

For the following supplement, let $\hat{\mathcal{E}}$ be the equivariant intersection cohomology sheaf of a refinement $\hat{\Delta}$ of Δ with refinement map $\pi : \hat{\Delta} \to \Delta$. The result is essential for a simplified proof of the Hard Lefschetz Theorem, see also [BreLu$_2$]:

Compatibility Theorem. *Let $\mathcal{E} \to \pi_*(\hat{\mathcal{E}})$ be a homomorphism of graded sheaves extending the identity $\mathcal{E}_o = \mathbf{R} = \pi_*(\hat{\mathcal{E}})_o$ at the zero cone o. Then the "global" intersection products are compatible, i.e., the following diagram is commutative:*

$$\begin{tikzcd} \mathcal{E}(\Delta) \times \mathcal{E}(\Delta, \partial\Delta) \arrow[rr] \arrow[dr] & & \hat{\mathcal{E}}(\hat{\Delta}) \times \hat{\mathcal{E}}(\hat{\Delta}, \partial\hat{\Delta}) \arrow[dl] \\ & A[-2n] \ . & \end{tikzcd}$$

The present article is a complete version of the results announced in [Fi]. – The authors gratefully acknowledge the hospitality of the *Institut de Mathématiques de Luminy* at Marseille, where part of this article has been written. For useful comments and remarks our thanks go to Tom Braden.

0. Preliminaries

For the convenience of the reader, we recall some basic notions, notations and constructions to be used in the sequel.

0.1 Let V be a real vector space of dimension n, and $A := S(V^*)$, the symmetric algebra on the dual vector space V^*, i.e., the algebra of real valued polynomial functions on V. We endow A with the even grading given by $A^2 = V^*$, a convention motivated by equivariant cohomology, and we let $\mathfrak{m} := A^{>0}$ be the homogeneous maximal ideal of A. For a graded A-module M, its reduction

$$\overline{M} \quad := \quad A/\mathfrak{m} \otimes_A M \,,$$

modulo $\mathfrak{m}$ is a graded real vector space.

For a strictly convex polyhedral cone $\sigma \subset V$, we let $V_\sigma \subset V$ denote its linear span. In analogy to the definition of A, we consider the graded algebra

$$A_\sigma \ := \ S(V_\sigma^*)\,.$$

We usually identify its elements with polynomial functions on the cone σ.

To avoid cumbersome notation, we admit graded homomorphisms even if they are not of degree zero.

0.2 Motivated by the coarse "toric topology" on a toric variety given by torus-invariant open sets, we consider a fan Δ (which need not be rational) in V as a finite topological space with the subfans as open subsets. The "affine" fans

$$\langle\sigma\rangle \ := \ \{\sigma\} \cup \partial\sigma \ \preceq \ \Delta \quad \text{with boundary fan} \quad \partial\sigma \ := \ \{\tau \in \Delta \ ; \ \tau \precneqq \sigma\}$$

form a basis of the fan topology by open sets that cannot be covered by smaller ones. Here $\preceq$ means that a cone is a face of another cone or that a set of cones is a subfan of some other fan.

Sheaf theory on that "fan space" is particularly simple since a presheaf given on the basis already "is" a sheaf. In particular, for a sheaf $\mathcal{F}$ on Δ, the equality

$$\mathcal{F}\big(\langle\sigma\rangle\big) \ = \ \mathcal{F}_\sigma$$

of the set of sections on the affine fan $\langle\sigma\rangle$ and the stalk at the point σ holds.

Furthermore, a sheaf $\mathcal{F}$ on Δ is flabby if and only if each restriction homomorphism $\varrho^\sigma_{\partial\sigma}\colon \mathcal{F}(\langle\sigma\rangle) \to \mathcal{F}(\partial\sigma)$ is surjective.

In particular, we consider (sheaves of) $\mathcal{A}$-modules, where $\mathcal{A}$ is the *structure sheaf* of Δ, i.e., the graded sheaf of polynomial algebras determined by $\mathcal{A}(\langle\sigma\rangle) := A_\sigma$, the restriction homomorphism $\varrho^\sigma_\tau\colon A_\sigma \to A_\tau$ being the restriction of functions on σ to the face $\tau \preceq \sigma$. The set of sections $\mathcal{A}(\Lambda)$ on a subfan $\Lambda \preceq \Delta$ constitutes the algebra of conewise polynomial functions on the support $|\Lambda|$ in a natural way.

Given a homomorphism $\varphi\colon \mathcal{F} \to \mathcal{F}'$ of sheaves on Δ and a subfan Λ, we often write

$$F_\Lambda \ := \ \mathcal{F}(\Lambda)\ , \quad F_\sigma \ := \ \mathcal{F}(\langle\sigma\rangle) \quad \text{and} \quad \varphi_\Lambda \ : \ F_\Lambda \to F'_\Lambda\ .$$

Similarly, for a pair of subfans (Λ, Λ_0) with $\Lambda_0 \preceq \Lambda$, we define

$$F_{(\Lambda,\Lambda_0)} \ := \ \ker(\varrho^\Lambda_{\Lambda_0}\colon F_\Lambda \to F_{\Lambda_0})\,,$$

the submodule of sections on Λ vanishing on Λ_0. In particular, for a purely n-dimensional subfan Λ (see 0.4 below), we consider the case that Λ_0 is the *boundary fan* $\partial\Lambda$, i.e., the subfan generated by those $(n-1)$-cones which are a facet of exactly one n-cone in Λ. The sections vanishing on $\partial\Lambda$ may be looked at as sections "with compact support".

0.3 Let $f\colon V \to W$ be a linear map inducing a map of fans between a fan Δ in V and a fan Λ in W, i.e., it maps each cone of Δ into a cone of Λ. Let $\mathcal{A}$ and $\mathcal{B}$ denote the corresponding sheaves of conewise polynomial functions, and let $\mathcal{F}$ on Δ and $\mathcal{G}$ on Λ be sheaves of graded $\mathcal{A}$- or $\mathcal{B}$-modules, respectively. For cones $\sigma \in \Delta$ and $\tau \in \Lambda$ with $f(\sigma) \subset \tau$, there is an induced homomorphism $B_\tau \to A_\sigma$ and thus, the structure of a B_τ-module on F_σ.

(a) The *direct image* $f_*(\mathcal{F})$ on Λ is the $\mathcal{B}$-module sheaf defined by

$$f_*(\mathcal{F})_\tau \;:=\; F_{f^{-1}(\tau)} \quad \text{with} \quad f^{-1}(\tau) \;:=\; \{\sigma \in \Delta; f(\sigma) \subset \tau\} \;\preceq\; \Delta\,.$$

The direct image of a flabby sheaf is again flabby.

(b) The *inverse image* $f^*(\mathcal{G})$ on Δ is the $\mathcal{A}$-module sheaf determined by

$$f^*(\mathcal{G})_\sigma \;:=\; A_\sigma \otimes_{B_\tau} G_\tau \quad \text{for } \sigma \in \Delta \text{ and the minimal } \tau \in \Lambda \text{ with } f(\sigma) \subset \tau.$$

We are especially interested in the following maps of fans:

(i) For a subdivision $\hat{\Delta}$ of Δ, the mapping $\mathrm{id}_V\colon (V,\hat{\Delta}) \to (V,\Delta)$ is such a morphism of fans. In particular, we will consider the case of an affine fan $\langle\sigma\rangle$ given by an n-dimensional cone, and its stellar subdivision

$$\hat{\sigma} := \partial\sigma + \lambda := \{\tau + \lambda; \tau \in \partial\sigma\}$$

with respect to a ray $\lambda := \ell \cap \sigma$, where ℓ is a one-dimensional linear subspace passing through the interior of σ.

(ii) For a cone $\sigma \in \Delta$, its closure in the fan topology is the star

$$\Delta_{\succeq\sigma} \;:=\; \{\gamma \in \Delta \;;\; \gamma \succeq \sigma\}\,.$$

In general, it is not a fan. The collection

$$\Delta/\sigma \;:=\; \{\pi(\gamma) \;;\; \gamma \in \Delta_{\succeq\sigma}\}$$

of the image cones with respect to the projection $\pi\colon V \to W := V/V_\sigma$, however, is a fan, called the *transversal fan* of σ with respect to Δ. The induced map $\pi_\sigma : \Delta_{\succeq\sigma} \to \Delta/\sigma$ is a homeomorphism.

(iii) Applying (ii) to the case of $\hat{\sigma}$ from (i), the projection $\pi : V \to W := V/\ell$ maps the boundary fan $\partial\sigma$ homeomorphically onto the "flattened boundary fan"

$$\Lambda_\sigma \;:=\; \hat{\sigma}/\lambda \;\cong\; \partial\sigma$$

in W. In that situation, choosing a linear form $T \in A^2$ with $T|_\lambda > 0$, we obtain isomorphisms $\ker(T) \cong W$ and thus $A \cong B[T]$, where we identify $B := S(W^*)$ with the subalgebra $\pi^*(B) \subset A$. – Moreover, for a sheaf $\mathcal{F}$ on $\langle\sigma\rangle$ and $\mathcal{G} := \pi_*(\mathcal{F}|_{\partial\sigma})$, there is a natural isomorphism of B-modules

$$G_{\Lambda_\sigma} \;\cong\; F_{\partial\sigma}\,. \tag{0.3.1}$$

0.4 We use notations as $\Delta^d := \{\gamma \in \Delta\ ;\ \dim \gamma = d\}$, $\Delta^{\le d}$, etc. The fan Δ in V is called:

(a) *oriented* if for each cone $\sigma \in \Delta$, an orientation or_σ of V_σ is fixed in such a way that orientations for full-dimensional cones coincide,

(b) *purely n-dimensional* if each maximal cone of Δ lies in Δ^n,

(c) *irreducible* if it is not the union of two proper subfans with intersection included in $\Delta^{\le n-2}$,

(d) *normal* if it is purely n-dimensional and for each cone $\sigma \in \Delta$, the transversal subfan Δ/σ is irreducible in V/V_σ , or, equivalently, if the support $|\Delta|$ is a normal pseudomanifold,

(e) *quasi-convex* if it is purely n-dimensional and the support $|\partial\Delta|$ of its boundary subfan is a real homology manifold. Note that a quasi-convex fan is normal, but not vice versa.

1. Pure Sheaves on a Fan

We recall the definition of the class of "pure" sheaves on a fan space that plays a key role in the sequel.

1.1 Definition. *A* ***pure sheaf*** *on a fan Δ is a flabby sheaf $\mathcal{F}$ of graded $\mathcal{A}$-modules such that, for each cone $\sigma \in \Delta$, the A_σ-module $F_\sigma = \mathcal{F}(\langle\sigma\rangle)$ is finitely generated and free.*

We collect some useful facts about these sheaves, proved in [BBFK$_2$] and [BreLu$_1$]:

Pure sheaves are built up from simple objects that correspond to the cones of the fan, or, equivalently, to the stalks of the structure sheaf. Up to a shift, such a simple sheaf is obtained from that stalk by a minimal extension process.

1.2 Simple Pure Sheaves. For each cone $\sigma \in \Delta$, we construct a "minimal" pure sheaf $\mathcal{L} =: {}_\sigma\mathcal{L}$ supported on the star $\Delta_{\succeq\sigma}$ and with stalk $L_\sigma = A_\sigma$ as follows: On the subfan $\Delta \setminus \Delta_{\succeq\sigma}$, we set $\mathcal{L} := 0$. By induction on the dimension, we extend it to the cones in $\Delta_{\succeq\sigma}$, starting with

$$L_\sigma \ := \ A_\sigma\,.$$

For a cone $\gamma \succeq \sigma$, we may assume that $L_{\partial\gamma}$ has been defined, and then set

$$L_\gamma \ := \ A_\gamma \otimes_{\mathbf{R}} \overline{L}_{\partial\gamma}\,.$$

The restriction homomorphism $\varrho^\gamma_{\partial\gamma}$ is defined by the following commutative diagram

$$\begin{array}{ccc}
L_\gamma := A_\gamma \otimes_{\mathbf{R}} \overline{L}_{\partial\gamma} & \longrightarrow & \overline{L}_{\partial\gamma} \\
\downarrow\ \varrho := \mathrm{id} \otimes s & \swarrow & \| \\
L_{\partial\gamma} = A_\gamma \otimes_{A_\gamma} L_{\partial\gamma} & \longrightarrow & \overline{L}_{\partial\gamma}
\end{array}$$

where the diagonal arrow $s\colon \overline{L}_{\partial\gamma} \to L_{\partial\gamma}$ is an $\mathbf{R}$-linear section of the reduction map in the bottom row.

1.3 Remarks. i) For each cone $\sigma \in \Delta$, the corresponding simple sheaf $\mathcal{L} := {}_\sigma\mathcal{L}$ is pure; it is characterized by the following properties:

a) $\overline{L}_\sigma \cong \mathbf{R}$,

b) for each cone $\tau \neq \sigma$, the reduced restriction homomorphism $\overline{L}_\tau \to \overline{L}_{\partial\tau}$ is an isomorphism.

In particular, property b) implies the vanishing of $\mathcal{L} := {}_\sigma\mathcal{L}$ outside of the star of σ.

ii) For the zero cone o, the "generic point" of Δ, the corresponding simple sheaf

$$\mathcal{E} \ := \ {}_\Delta\mathcal{E} \ := \ {}_o\mathcal{L}$$

is called the *equivariant intersection cohomology sheaf* (or the minimal extension sheaf) of Δ. For a quasi-convex fan Δ, we may define its (virtual) *intersection cohomology* as

$$IH(\Delta) \ := \ \overline{E}_\Delta \ . \tag{1.3.1}$$

iii) By extending scalars, each "local" sheaf ${}_\sigma\mathcal{L}$ is derived from the "global" sheaf ${}_{\Delta/\sigma}\mathcal{E}$ of the corresponding transversal fan: As in 0.3 (ii), we let $\pi_\sigma := \pi|_{\Delta_{\succeq\sigma}} : \Delta_{\succeq\sigma} \to \Delta/\sigma$ denote the homeomorphism induced from the projection $V \to V/V_\sigma$. The inverse image $\pi_\sigma^*({}_{\Delta/\sigma}\mathcal{E})$ is a flabby sheaf of graded $\mathcal{A}$-modules on the closed subset $\Delta_{\succeq\sigma}$ of Δ. Its trivial extension to the whole fan space Δ then yields the sheaf ${}_\sigma\mathcal{L}$.

The following elementary decomposition theorem has been proved in [BBFK$_2$, 2.4] and in [BreLu$_1$, 5.3]:

1.4 Decomposition Theorem. *Every pure sheaf $\mathcal{F}$ on Δ admits a natural direct sum decomposition of $\mathcal{A}$-modules*

$$\mathcal{F} \quad \cong \quad \bigoplus_{\sigma\in\Delta} \left({}_\sigma\mathcal{L} \otimes_{\mathbf{R}} K_\sigma\right)$$

with $K_\sigma := K_\sigma(\mathcal{F}) := \ker\left(\overline{\varrho}^\sigma_{\partial\sigma}\colon \overline{F}_\sigma \to \overline{F}_{\partial\sigma}\right)$, a finite dimensional graded vector space.

For a proof of the following application, we refer to [BBFK$_2$, 2.5].

1.5 Example. Let $\varphi := \mathrm{id}_V : (V, \hat{\Delta}) \to (V, \Delta)$ be a refinement. Then $\varphi_*(\hat{\mathcal{E}})$ is a pure sheaf. Its decomposition is of the form

$$\varphi_*(\hat{\mathcal{E}}) \quad \cong \quad \mathcal{E} \oplus \bigoplus_{\sigma\in\Delta^{\geq 2}} \left({}_\sigma\mathcal{L} \otimes_{\mathbf{R}} K_\sigma\right),$$

where the K_σ now are (strictly) positively graded vector spaces and the "correction terms" are supported on the closed subset $\Delta^{\geq 2}$.

1.6 Remark. For a pure sheaf $\mathcal{F}$ on the boundary fan $\partial\sigma$ of an n-dimensional cone and the projection mapping $\pi\colon (V, \partial\sigma) \to (W, \Lambda_\sigma)$ corresponding to a ray λ as in 0.3, (ii), the direct image $\pi_*(\mathcal{F}|_{\partial\sigma})$ is a pure sheaf on Λ_σ. ◻

2. The Dual of a Pure Sheaf

In this section, the symbol $\mathcal{F}$ always denotes a pure sheaf on an oriented fan Δ. Furthermore, unless otherwise stated, the symbol Hom is understood to mean Hom_A, and $\otimes$ means $\otimes_{\mathbf{R}}$. Moreover, for a cone $\sigma \in \Delta$, we consider $\det V_\sigma^* := \bigwedge^{\dim\sigma} V_\sigma^*$ as a graded vector space concentrated in degree $2\dim\sigma$, with the convention $\det V_o^* = \mathbf{R}$.

To $\mathcal{F}$, we associate its dual $\mathcal{DF}$ and show the following properties: The dual is again a pure sheaf on Δ, and for each normal subfan Λ, the module of sections $(\mathcal{DF})_\Lambda$ is the dual of the module $F_{(\Lambda,\partial\Lambda)}$ of sections with compact supports of $\mathcal{F}$.

2.1 Construction of the dual sheaf. To construct the dual $\mathcal{DF}$ of the pure sheaf $\mathcal{F}$ on Δ, we first define its sections over affine fans in such a way that duality holds by definition.

Sections over a cone $\sigma \in \Delta$. As A_σ-module, we define $(\mathcal{DF})_\sigma = \mathcal{DF}(\langle\sigma\rangle)$ by

$$(2.1.1) \qquad (\mathcal{DF})_\sigma \quad := \quad \mathrm{Hom}(F_{(\sigma,\partial\sigma)}, A_\sigma) \otimes \det V_\sigma^* \ .$$

Restriction homomorphisms. The homomorphism ϱ^σ_τ for $\sigma \succeq \tau$ is constructed in two steps: In the first step, we deal with the case of a facet; in the second step, we extend this recursively to the general situation of a face of arbitrary codimension.

To that end, we need *transition coefficients* $\varepsilon^\sigma_\tau = \pm 1$ for the facets τ of σ: For $d := \dim\sigma$, there exists a natural map $\kappa\colon \bigwedge^{d-1} V_\sigma^* \to \bigwedge^{d-1} V_\tau^* = \det V_\tau^*$. We choose a linear form h on V_σ^* with $V_\tau = \ker(h)$ and $h|_\sigma \geq 0$. Every element of $\det V_\sigma^*$ decomposes in the form $h \wedge \eta$ with unique image $\kappa(\eta)$. We thus obtain a homomorphism

$$(2.1.2) \qquad \psi_h : \det V_\sigma^* \ \longrightarrow\ \det V_\tau^* \ , \quad h\wedge\eta \ \longmapsto\ \kappa(\eta) \ .$$

If now $\omega_\sigma \in \det V_\sigma^*$ and $\omega_\tau \in \det V_\tau^*$ define the orientations of σ respectively τ, we set

$$(2.1.3) \qquad \varepsilon^\sigma_\tau := \begin{cases} +1 & \text{if } \psi_h(\omega_\sigma) \in \mathbf{R}_{>0}\ \omega_\tau \ , \\ -1 & \text{otherwise.} \end{cases}$$

Step 1: Restriction homomorphism for a facet $\tau \prec_1 \sigma$. Using again the linear form $h \in V_\sigma^*$, we are going to define another homomorphism

$$(2.1.4) \qquad \varphi_h : \mathrm{Hom}(F_{(\sigma,\partial\sigma)}, A_\sigma) \ \longrightarrow\ \mathrm{Hom}(F_{(\tau,\partial\tau)}, A_\tau)$$

and see that

$$\varphi_{\lambda h} = \lambda\varphi_h \quad \text{and} \quad \psi_{\lambda h} = \lambda^{-1}\psi_h \tag{2.1.5}$$

for every non-zero scalar $\lambda \in \mathbf{R}$. Thus the homomorphism

$$\varphi_h \otimes \psi_h : \operatorname{Hom}(F_{(\sigma,\partial\sigma)}, A_\sigma) \otimes \det V_\sigma^* \longrightarrow \operatorname{Hom}(F_{(\tau,\partial\tau)}, A_\tau) \otimes \det V_\tau^*$$

does not depend on the special choice of h, and we may set

$$\varrho_\tau^\sigma := \varepsilon_\tau^\sigma \cdot \varphi_h \otimes \psi_h \ .$$

The map φ_h associates to a homomorphism $f\colon F_{(\sigma,\partial\sigma)} \to A_\sigma$ the homomorphism $\varphi_h(f)\colon F_{(\tau,\partial\tau)} \to A_\tau$, which acts in the following way: We first extend a section $s \in F_{(\tau,\partial\tau)}$ trivially to $\partial\sigma$ and then to a section $\check{s} \in F_\sigma$; we thus have $h\check{s} \in F_{(\sigma,\partial\sigma)}$ and may finally set

$$\varphi_h(f)(s) := f(h\check{s})|_\tau \in A_\tau \ . \tag{2.1.6}$$

In order to see that this definition is independent of the particular choice of $\check{s}$, we present an alternative description, following the argument on p.36 of [BBFK]: We use three exact sequences, starting with

$$0 \to F_{(\sigma,\partial\sigma)} \longrightarrow F_\sigma \longrightarrow F_{\partial\sigma} \longrightarrow 0 \ . \tag{2.1.7}$$

The second one is composed of the multiplication with h and the projection onto the cokernel:

$$0 \longrightarrow A_\sigma \xrightarrow{\mu_h} A_\sigma \longrightarrow A_\tau \longrightarrow 0 \ . \tag{2.1.8}$$

Eventually the subfan $\partial_\tau\sigma := \partial\sigma \setminus \{\tau\}$ of $\partial\sigma$ gives rise to the exact sequence

$$0 \to F_{(\tau,\partial\tau)} \to F_{\partial\sigma} \to F_{\partial_\tau\sigma} \to 0 \ . \tag{2.1.9}$$

The associated Hom-sequences provide a diagram

(2.1.10)

$$\begin{tikzcd}
 & & & \operatorname{Ext}(F_{\partial_\tau\sigma}, A_\sigma) \arrow[d] & \\
 & \operatorname{Hom}(F_\sigma, A_\sigma) \arrow[r] & \operatorname{Hom}(F_{(\sigma,\partial\sigma)}, A_\sigma) \arrow[r, "\alpha"] & \operatorname{Ext}(F_{\partial\sigma}, A_\sigma) \arrow[d, "\beta"] & \\
\operatorname{Hom}(F_{(\tau,\partial\tau)}, A_\sigma) \arrow[r] & & \operatorname{Hom}(F_{(\tau,\partial\tau)}, A_\tau) \arrow[r, "\gamma"] & \operatorname{Ext}(F_{(\tau,\partial\tau)}, A_\sigma) \arrow[r] & \operatorname{Ext}(F_{(\tau,\partial\tau)}, A_\sigma)
\end{tikzcd}$$

with $\operatorname{Ext} = \operatorname{Ext}_A^1$. We show that γ is an isomorphism; we then have

$$\varphi_h := \gamma^{-1} \circ \beta \circ \alpha \,. \tag{2.1.11}$$

In fact, the rightmost arrow in the bottom row is the zero homomorphism, since it is induced by multiplication with h, which annihilates $F_{(\tau,\partial\tau)}$. On the other hand, the A_τ-module $F_{(\tau,\partial\tau)}$ is a torsion module over A_σ, so that $\operatorname{Hom}(F_{(\tau,\partial\tau)}, A_\sigma)$ vanishes.

Step 2: Restriction homomorphism for faces of higher codimension. For a face $\tau \prec \sigma$ of codimension $r \geq 2$, we choose a "flag"

$$\tau =: \tau_0 \quad \prec_1 \quad \tau_1 \quad \prec_1 \quad \ldots \quad \prec_1 \quad \tau_r := \sigma$$

of relative facets joining τ and σ. Defining the restriction homomorphism ϱ_τ^σ as the composition of the $\varrho_{\tau_i}^{\tau_{i+1}}$, we have to show that the result does not depend on the particular choice of the flag. This is easy to see in the case $r = 2$: For two flags $\gamma \prec_1 \tau \prec_1 \sigma$ and $\gamma \prec_1 \tau' \prec_1 \sigma$ and $h, h' \in V_\sigma^*$ as above, we set $g := h|_{V_{\tau'}}$ and $g' := h'|_{V_\tau}$, and then find

$$\varphi_{g'} \circ \varphi_h \quad = \quad \varphi_g \circ \varphi_{h'} \quad \text{and} \quad \psi_{g'} \circ \psi_h \quad = \quad -\psi_g \circ \psi_{h'} \ ,$$

whence

$$\varrho_\gamma^\tau \circ \varrho_\tau^\sigma \quad = \quad \varrho_\gamma^{\tau'} \circ \varrho_{\tau'}^\sigma \ . \tag{2.1.12}$$

Thus, for general r, it suffices to verify that every two such flags can be transformed into each other in such a way that in each step, only one intermediate cone is replaced by another one. We proceed by induction on the codimension r.

To prove that claim, we may assume $\tau = o$ (otherwise, we replace Δ with Δ/τ) and $\Delta = \langle\sigma\rangle$. We want to compare the given flag with a second one, say $o \prec_1 \tilde{\tau}_1 \prec_1 \tilde{\tau}_2 \prec_1 \ldots \prec_1 \tilde{\tau}_r = \sigma$. There is a chain of rays $\varrho_1 := \tau_1, \ldots, \varrho_s := \tilde{\tau}_1$ such that the two-dimensional cones $\varrho_i + \varrho_{i+1}$ belong to Δ. We now proceed by a second induction on s. For $s = 1$, we may pass to the fan Δ/τ_1 and use the first induction hypothesis for $r-1$. For the induction step, it evidently suffices to consider the case $s = 2$. Choosing any auxiliary flag of the form $o \prec_1 \tau_1 \prec_1 \tau_1 + \tilde{\tau}_1 \prec_1 \ldots \prec_1 \sigma$, the case $s = 1$ yields its equivalence with the start flag. On the other hand, by (2.1.12), the auxiliary flag is equivalent to the one obtained by interchanging τ_1 and $\tilde{\tau}_1$, and this in turn is equivalent to the "twiddled" flag. $\square$

We now show that the formula (2.1.1) for cones extends even to normal subfans. To that end, we need the following preparatory results:

2.2 Lemma. (i) *For an arbitrary fan Δ, there is a natural isomorphism*

$$\Theta : \bigoplus_{\sigma \in \Delta^n} (\mathcal{DF})_\sigma \quad \xrightarrow{\cong} \quad \operatorname{Hom}(F_{(\Delta, \Delta^{\leq n-1})}, A) \otimes \det V^* \ . \tag{2.2.1}$$

(ii) *If Δ is purely n-dimensional, the A-modules*

$$F_{(\Delta, \Delta^{\leq n-1})} \subset F_{(\Delta, \partial\Delta)} \subset F_\Delta$$

are torsion-free and of the same rank. As a consequence, the restriction homomorphisms

$$\operatorname{Hom}(F_\Delta, A) \longrightarrow \operatorname{Hom}(F_{(\Delta,\partial\Delta)}, A) \longrightarrow \operatorname{Hom}(F_{(\Delta, \Delta^{\leq n-1})}, A)$$

for the dual modules are injective.

(iii) *In the setup of cellular ("Čech") cochains and cocycles as in section 3 of* [BBFK$_2$], *for an arbitrary sheaf* $\mathcal{G}$ *on* Δ, *there is an isomorphism*

$$Z^0(\Delta;\mathcal{G}) \,=\, G_{(\Delta,\partial\Delta)}\,, \tag{2.2.2}$$

and for a normal fan, we also have an isomorphism

$$Z^0(\Delta,\partial\Delta;\mathcal{G}) \,=\, G_\Delta\,. \tag{2.2.3}$$

Proof: (i) For each n-dimensional cone σ, the equality $\det V_\sigma^* = \det V^*$ holds. Hence, the isomorphism Θ is immediately obtained from the defining equality (2.1.1) by applying the additive functor $\mathrm{Hom}(\_\,,A)\otimes V^*$ to the obvious direct sum decomposition

$$F_{(\Delta,\Delta^{\le n-1})} \,\cong\, \bigoplus_{\sigma\in\Delta^n} F_{(\sigma,\partial\sigma)}\;.$$

(ii) For the special case $\mathcal{F}=\mathcal{E}$, this has been proved in [BBFK$_2$, 6.1, i)]. The proof clearly carries over to arbitrary pure sheaves.

(iii) We recall that the submodule

$$Z^0(\Delta,\partial\Delta;\mathcal{G}) \,\subset\, C^0(\Delta,\partial\Delta;\mathcal{G}) \,=\, \bigoplus_{\sigma\in\Delta^n} G_\sigma$$

of degree zero cocycles $g=(g_\sigma)$ relative to $\partial\Delta$ consists of those cochains that satisfy $g_\sigma|_\tau = g_{\sigma'}|_\tau$ whenever $\tau\in\Delta^{n-1}$ is a common facet of two n-cones σ and σ', whereas for the submodule

$$Z^0(\Delta;\mathcal{G}) \,\subset\, Z^0(\Delta,\partial\Delta;\mathcal{G})$$

of absolute cocycles, we require in addition that the restriction of g_σ to each "outer" facet $\tau\in\partial\Delta^{n-1}$ vanishes.

For (2.2.2) and (2.2.3) we note that the right hand side is always contained in the left hand side. In order to see the reverse inclusion, we have to show that $g_\sigma|_\gamma = g_{\sigma'}|_\gamma$ holds whenever $\gamma\in\Delta$ is a common face of two n-cones σ and σ'. Since Δ is normal, the cones σ and σ' can be joined by a chain of n-cones intersecting successively in common facets containing γ. It thus suffices to consider the case that γ is a facet, where the statement is obvious. □

2.3 Theorem. *For a normal oriented fan* Δ, *the natural isomorphism* Θ *of* (2.2.1) *induces isomorphisms*

$$(\mathcal{DF})_\Delta \;\xrightarrow{\cong}\; \mathrm{Hom}(F_{(\Delta,\partial\Delta)},A)\otimes\det V^* \tag{2.3.1}$$

and

$$(\mathcal{DF})_{(\Delta,\partial\Delta)} \;\xrightarrow{\cong}\; \mathrm{Hom}(F_\Delta,A)\otimes\det V^*\;. \tag{2.3.2}$$

Proof: Using the formalism of Čech cochains as in (iii) of the Lemma, we restate the assertion as follows: For any 0-cochain $\psi=(\psi_\sigma)\in\bigoplus_{\sigma\in\Delta^n}\mathcal{DF}_\sigma$ and its image

$\Theta(\psi)$ in $\mathrm{Hom}(F_{(\Delta,\Delta^{\le n-1})}, A) \otimes \det V^*$, the equivalences

$$\psi \in Z^0(\Delta, \partial\Delta; \mathcal{DF}) \iff \Theta(\psi) \in \mathrm{Hom}(F_{(\Delta,\partial\Delta)}, A) \otimes \det V^* \tag{2.3.3}$$

and

$$\psi \in Z^0(\Delta; \mathcal{DF}) \iff \chi = \Theta(\psi) \in \mathrm{Hom}(F_\Delta, A) \otimes \det V^* \tag{2.3.4}$$

hold.

To prove these equivalences, we choose an auxiliary function $h = \prod_{i=1}^r h_i \in A^{2r}$ as the lowest degree product of linear forms h_i that vanishes on $\bigcup_{\tau\in\Delta^{n-1}} V_\tau$; so each V_τ is the kernel V_i of some h_i. After fixing a positive volume form on V and thus, an isomorphism $\mathbf{R} \xrightarrow{\cong} \det V^*$, the homomorphisms $\psi_{h_i}$of (2.1.2) provide isomorphisms $\mathbf{R} \cong \det V^* \cong \det V_i^*$. We may thus drop the determinant factors on the right hand side, and for each cone $\gamma \in \Delta^{\ge n-1}$, we may replace $(\mathcal{DF})_\gamma$ with $\mathrm{Hom}(F_{(\gamma,\partial\gamma)}, A_\gamma)$ and the restriction maps with $\pm\varphi_{h_i}$. Using the obvious inclusions

$$hF_{(\Delta,\partial\Delta)} \subset hF_\Delta \subset F_{(\Delta,\Delta^{\le n-1})}$$

of torsion-free A-modules, the right hand sides of (2.3.3) and (2.3.4) are equivalent to the inclusions

$$\chi(hF_{(\Delta,\partial\Delta)}) \subset hA \quad \text{and} \quad \chi(hF_\Delta) \subset hA, \quad \text{where} \quad \chi := \Theta(\varphi)\ .$$

"$\Longrightarrow$": In order to prove these implications, it suffices to show that for a pertinent 0-cocycle ψ, the divisibility relation $h_i \mid \chi(hf)$ holds for each index i and for an arbitrary section f in $F_{(\Delta,\partial\Delta)}$ or F_Δ, respectively.

With $f_\sigma := f|_\sigma \in F_\sigma$ for an n-cone σ, we write

$$\chi(hf) = \sum_{\sigma\in\Delta^n} \psi_\sigma(hf_\sigma) \in A\,.$$

For each index $i = 1, \dots, r$, we introduce the monomial $g_i := h/h_i \in A^{2r-2}$. For the implication in (2.3.3), we consider a relative 0-cocycle $\psi = (\psi_\sigma) \in Z^0(\Delta, \partial\Delta; \mathcal{DF})$ and a section $f \in F_{(\Delta,\partial\Delta)}$ "with compact support". If $\sigma \cap V_i$ is not a facet or belongs to $\partial\Delta$, then $g_i f_\sigma$ lies in $F_{(\sigma,\partial\sigma)}$ and thus $\psi_\sigma(hf_\sigma) = h_i\psi_\sigma(g_i f_\sigma) \in h_i A$ holds. Otherwise, there is precisely one n-cone $\sigma' \ne \sigma$ such that $\tau := \sigma \cap V_i$ is a common facet of both, σ and σ'. We now verify that h_i divides the sum $\psi_\sigma(hf_\sigma) + \psi_{\sigma'}(hf_{\sigma'})$ or, equivalently, that

$$\psi_\sigma(hf_\sigma)\,|_{V_i} = -\psi_{\sigma'}(hf_{\sigma'})\,|_{V_i} \tag{2.3.5}$$

holds. Using the extension $g_i f_\sigma \in F_{(\sigma,\partial_\tau\sigma)}$ of $(g_i f)|_\tau \in F_{(\tau,\partial\tau)}$ in formula (2.1.6), we obtain

$$\psi_\sigma(hf_\sigma)|_{V_i} = \big(\varphi_{h_i}(\psi_\sigma)\big)\big((g_i f_\sigma)|_\tau\big)\ .$$

By the relative cocycle condition, ψ_σ and $\psi_{\sigma'}$ restrict to the same section in $(\mathcal{DF})_\tau$. According to the choice of the transition coefficients ε_τ^σ in the definition of the restriction homomorphism ϱ_τ^σ in (2.1.8), that yields

$$\varphi_{h_i}(\psi_\sigma) = -\varphi_{h_i}(\psi_{\sigma'})\ ,$$

which implies our claim. If $\psi \in Z^0(\Delta;\mathcal{DF})$ is an absolute cocycle and $f \in F_\Delta$, the argument is as above, only in the case that $\tau := \sigma \cap V_i$ is an "outer" facet of σ (i.e., contained in $\partial\Delta$), one has to use the fact that $\psi_\sigma|_\tau = 0$.

"$\Longleftarrow$": For this implication, we assume that $\chi = \Theta(\psi) : F_{(\Delta,\Delta^{\le n-1})} \to A$ is a homomorphism which can be extended to the larger modules $F_{(\Delta,\partial\Delta)}$ or F_Δ, respectively. We have to show the pertinent cocycle condition for $\psi = (\psi_\sigma)$, namely, the equality $\psi_\sigma|_\tau = \psi_{\sigma'}|_\tau$ whenever τ is a common "inner" facet of two n-cones $\sigma, \sigma' \in \Delta$, and in the second ("absolute") case, the vanishing $\psi_\sigma|_\tau = 0$ if τ is an "outer" facet of σ.

Let i be the index with $\ker(h_i) = V_\tau$. We fix an arbitrary section $f_0 \in F_{(\tau,\partial\tau)}$ and, as for (2.1.6), extend it to sections $f \in F_\sigma$, $f' \in F_{\sigma'}$ vanishing on all the remaining facets of σ and of σ', respectively. Patching them together and extending by 0 yields a section $f_1 \in F_{(\Delta,\partial\Delta)}$. Then the equation

$$h_i\chi(f_1) \;=\; \chi(h_i f_1) \;=\; \chi(h_i f + h_i f') \;=\; \psi_\sigma(h_i f) + \psi_{\sigma'}(h_i f') \ ,$$

after restriction to τ, yields

$$0 \;=\; \big(h_i\chi(f_1)\big)\Big|_\tau \;=\; \varphi_{h_i}(\psi_\sigma)(f_0) + \varphi_{h_i}(\psi_{\sigma'})(f_0) \;=\; (\psi_\sigma|_\tau - \psi_{\sigma'}|_\tau)(f_0) \ .$$

Finally, we leave it to the reader to consider the remaining case where $\chi \in \mathrm{Hom}(F_\Delta, A)$ and $\tau \in \partial\Delta$. □

2.4 Theorem: *The dual sheaf $\mathcal{DF}$ of a pure sheaf $\mathcal{F}$ is again pure.*

Proof: As in Corollary 4.12 in [BBFK], the A_σ-module $F_{(\sigma,\partial\sigma)}$ is free and thus also its dual $(\mathcal{DF})_\sigma$; hence, we only have to prove that, for each cone $\sigma \in \Delta$, the restriction homomorphism

$$\varrho^\sigma_{\partial\sigma} : (\mathcal{DF})_\sigma \longrightarrow (\mathcal{DF})_{\partial\sigma}$$

is surjective.

To that end, we first interpret $(\mathcal{DF})_{\partial\sigma}$. We may assume $\dim\sigma = n$ and use the setup of 0.3 (iii). For $\mathcal{G} := \pi_*(\mathcal{F}|_{\partial\sigma})$, as in (0.3.a), there is a natural isomorphism

$$(\mathcal{DG})_{\Lambda_\sigma} \;\cong\; (\mathcal{DF})_{\partial\sigma} \tag{2.4.1}$$

of B-modules, while for the complete fan Λ_σ in W, Theorem 2.3 yields

$$(\mathcal{DG})_{\Lambda_\sigma} \;\cong\; \mathrm{Hom}_B(G_{\Lambda_\sigma}, B) \otimes \det W^* \ .$$

Using the isomorphism (0.3.1), we thus obtain a chain of isomorphisms

$$(\mathcal{DF})_{\partial\sigma} \cong (\mathcal{DG})_{\Lambda_\sigma} \cong \mathrm{Hom}_B(G_{\Lambda_\sigma}, B) \;\cong\; \mathrm{Hom}_B(F_{\partial\sigma}, B) \ . \tag{2.4.2}$$

Eventually, using these isomorphisms, a section $\beta \in (\mathcal{DF})_{\partial\sigma}$ may be interpreted as an element of $\mathrm{Hom}_B(F_{\partial\sigma}, B)$.

To proceed with the proof, we introduce the sheaf $\mathcal{H} := \pi^*(\mathcal{G})$ on $\hat{\sigma}$. There are isomorphisms

$$(2.4.3) \qquad H_{\hat{\sigma}} \;\cong\; A \otimes_B G_{\Lambda_\sigma} \;\cong\; A \otimes_B F_{\partial\sigma} \quad \text{and} \quad H_{\partial\hat{\sigma}} \;\cong\; F_{\partial\sigma}$$

and a "Thom isomorphism"

$$(2.4.4) \qquad \mu_g \,:\, H_{\hat{\sigma}} \xrightarrow{\cong} gH_{\hat{\sigma}} \;=\; H_{(\hat{\sigma},\partial\hat{\sigma})} \;,\; h \mapsto gh$$

with a conewise linear function $g \in A^2_{(\hat{\sigma},\partial\hat{\sigma})}$, unique up to a non-zero scalar multiple, that is constructed conewise as follows: We fix a nontrivial linear form $f \in A^2_\lambda$. For a facet $\tau \prec_1 \sigma$, let $g_\tau \in A^2$ be the unique linear form with $\ker(g_\tau) = V_\tau$ and $g_\tau|_\lambda = f$. Then we set $g|_{\tau+\lambda} := g_\tau$.

For each facet τ of σ, the function g_τ induces an isomorphism

$$\det V^* \;\cong\; \det V^*_\tau \;,\; g_\tau \wedge \eta \;\mapsto\; \eta|_{V_\tau} \;.$$

Then the composed isomorphism $\det V^* \cong \det V^*_\tau \cong \det W^*$ is independent of τ. We thus may drop the determinant factors.

We want to show that an inverse image $\alpha \in (\mathcal{DF})_\sigma = \operatorname{Hom}(F_{(\sigma,\partial\sigma)}, A)$ of $\beta \in (\mathcal{DF})_{\partial\sigma}$ with respect to $\varrho^\sigma_{\partial\sigma}$ is given by the composition

$$F_{(\sigma,\partial\sigma)} \xrightarrow{\;i\;} H_{(\hat{\sigma},\partial\hat{\sigma})} \xrightarrow{\mu_{g^{-1}}} H_{\hat{\sigma}} \;\cong\; A \otimes_B F_{\partial\sigma} \xrightarrow{\mathrm{id}_A \otimes \beta} A \otimes_B B \;=\; A$$

where $\mu_{g^{-1}}$ is the isomorphism "division by g" corresponding to (2.4.4), and the homomorphism i is constructed as follows: Since F_σ is a free A-module and the restriction homomorphism $H_{\hat{\sigma}} \to H_{\partial\hat{\sigma}}$ is surjective, cf. (2.4.3), the operator $\varrho^\sigma_{\partial\sigma}$ for the sheaf $\mathcal{F}$ admits a factorization of the form

$$F_\sigma \xrightarrow{\;j\;} H_{\hat{\sigma}} \twoheadrightarrow H_{\partial\hat{\sigma}} \;\cong\; F_{\partial\sigma} \,.$$

Since $j(F_{(\sigma,\partial\sigma)})$ clearly is included in $H_{(\hat{\sigma},\partial\hat{\sigma})}$, we may choose $i := j|_{F_{(\sigma,\partial\sigma)}}$.

To prove the equality $\alpha|_{\partial\sigma} = \beta$, it still remains to show that $\alpha|_\tau = \beta|_\tau$ for all facets $\tau \prec_1 \sigma$. Here we identify the naturally isomorphic algebras B and A_τ.

We fix an arbitrary section $s \in F_{(\tau,\partial\tau)} \subset F_{\partial\sigma}$, where the inclusion is given by trivial extension. Using the isomorphisms (2.4.3) and (2.4.4), any further extension $\check{s}$ of s to a section of $\mathcal{F}$ on the whole cone σ, looked at as section in $H_{\hat{\sigma}} \supset F_\sigma$, can be written in the form

$$\check{s} \;=\; 1 \otimes s + gd \;\in\; H_{\hat{\sigma}} \;\cong\; A \otimes_B F_{\partial\sigma}$$

with some correction term $d \in H_{\hat{\sigma}}$. Recalling the formula (2.1.6) in the definition of the homomorphism ϱ^σ_τ for $\mathcal{DF}$, we have to show that the restriction of the polynomial function $\alpha(g_\tau \cdot \check{s}) \in A_\sigma$ to τ coincides with $\beta(s)$. To that end, we note that $g_\tau \cdot (1 \otimes s) = g \cdot (1 \otimes s)$ holds, since the support of $1 \otimes s \in H_{\hat{\sigma}}$ is contained in $\tau + \lambda$. So we eventually have the equality

$$\alpha(g_\tau \cdot \check{s})|_\tau \;=\; (\mathrm{id}_A \otimes \beta)(1 \otimes s + g_\tau d)|_\tau \;=\; (\mathrm{id}_A \otimes \beta)(1 \otimes s)|_\tau \;,$$

and thus $\varrho_\tau^\sigma(\alpha)$ maps s to $\beta(s)$. $\square$

In order to see that the dual sheaf $\mathcal{DF}$ of a simple pure sheaf again is simple, we need biduality:

2.5 Biduality Theorem. [BreLu$_1$, 6.23] *Every pure sheaf on an oriented fan is reflexive: For such a sheaf $\mathcal{F}$, there exists a natural isomorphism*

$$\mathcal{F} \xrightarrow{\cong} \mathcal{D}(\mathcal{DF}) .$$

Proof: Over a cone $\sigma \in \Delta$, the biduality isomorphism $F_\sigma \to \mathcal{DDF}_\sigma$ is obtained using these isomorphisms:

$$\begin{aligned}(\mathcal{DDF})_\sigma &= \operatorname{Hom}\big((\mathcal{DF})_{(\sigma,\partial\sigma)}, A_\sigma\big) \otimes \det V_\sigma^* \\ &\cong \operatorname{Hom}\big(\operatorname{Hom}(F_\sigma, A_\sigma) \otimes \det V_\sigma^*, A_\sigma\big) \otimes \det V_\sigma^* \\ &\cong \operatorname{Hom}\big(\operatorname{Hom}(F_\sigma, A_\sigma) \otimes \det V_\sigma^*, A_\sigma \otimes \det V_\sigma^*\big) \\ &\cong \operatorname{Hom}\big(\operatorname{Hom}(F_\sigma, A_\sigma), A_\sigma\big)\end{aligned} \qquad (2.5.1)$$

where the first isomorphism follows from Theorem 2.3 with the fan $\Delta := \langle\sigma\rangle$ in the vector space V_σ. The free A_σ-module F_σ is reflexive, so it can be naturally identified with the fourth module in (2.5.1). Since this conewise construction is natural, it carries over to the sheaves. $\square$

2.6 Corollary. [BreLu$_1$, 6.26] *For each cone $\sigma \in \Delta$, the simple pure sheaf ${}_\sigma\mathcal{L}$ satisfies*

$$\mathcal{D}({}_\sigma\mathcal{L}) \;\cong\; {}_\sigma\mathcal{L} \otimes \det V_\sigma^* \;.$$

In particular, the equivariant intersection cohomology sheaf $\mathcal{E}$ is self-dual with an isomorphism

$$\vartheta : \mathcal{E} \xrightarrow{\cong} \mathcal{DE} \qquad (2.6.1)$$

of degree zero.

Proof: Clearly, by biduality, $\mathcal{DF} = 0$ implies $\mathcal{F} = 0$. On the other hand, the duality functor respects a direct sum decomposition of pure sheaves. Since the bidual $\mathcal{D}\big(\mathcal{D}({}_\sigma\mathcal{L})\big) \cong {}_\sigma\mathcal{L}$ is simple, the Decomposition Theorem 1.4 implies that the dual $\mathcal{D}({}_\sigma\mathcal{L})$ must be a simple sheaf. For a pure sheaf $\mathcal{F}$ and a cone $\sigma \in \Delta$, the A_σ-module $F_{\partial\sigma}$ is a torsion module, whence $F_\sigma = 0$ if and only if $F_{(\sigma,\partial\sigma)} = 0$. Hence a pure sheaf and its dual have the same support, so $\mathcal{D}({}_\sigma\mathcal{L})$ and ${}_\sigma\mathcal{L}$ agree up to a shift. To determine it explicitly, we use the equality ${}_\sigma L_{(\sigma,\partial\sigma)} = {}_\sigma L_\sigma = A_\sigma$, which yields $D({}_\sigma\mathcal{L})_\sigma \cong \operatorname{Hom}(A_\sigma, A_\sigma) \otimes \det V_\sigma^* \cong A_\sigma \otimes \det V_\sigma^*$. $\square$

3. The Intersection Product

In order to make precise the naturality of the intersection product we need this notion:

3.1 Definition. *A **duality correlation** on Δ is a sheaf homomorphism*

$$\varphi : \mathcal{E} \longrightarrow \mathcal{D}\mathcal{E}$$

of degree 0 *from the equivariant intersection cohomology sheaf to its dual extending the natural identification*

$$\mathcal{E}_o = \mathbf{R} \xrightarrow{1 \mapsto 1^*} \mathbf{R}^* = \mathcal{D}\mathcal{E}_o \ .$$

After multiplication with an appropriate scalar factor if necessary, any isomorphism $\mathcal{E} \to \mathcal{D}\mathcal{E}$ is such a duality correlation. We aim at the following result:

3.2 Theorem. *On every fan Δ, there is a unique duality correlation. It defines a self duality $\mathcal{E} \cong \mathcal{D}\mathcal{E}$ for the equivariant intersection cohomology sheaf $\mathcal{E}$.*

Existence has already been shown in 2.6. Before proving uniqueness, we first use the correlation to introduce an intersection product.

3.3 Remark and Definition. Let Δ be a normal n-dimensional oriented fan. If we fix a positive volume form $\omega \in \det V^*$, then every duality correlation φ gives rise to an *intersection product* on Δ, i.e., a pairing

(PD) $$E_\Delta \times E_{(\Delta,\partial\Delta)} \longrightarrow A[-2n]$$

as follows: The isomorphism Θ of (2.3.1) yields an isomorphism

(Dω) $$(\mathcal{D}\mathcal{E})_\Delta \xrightarrow{\Theta_V} \mathrm{Hom}\big(E_{(\Delta,\partial\Delta)}, A\big) \otimes \det V^* \xrightarrow{\cong} \mathrm{Hom}(E_{(\Delta,\partial\Delta)}, A[-2n]) \ ;$$

its composition with the duality correlation φ_Δ on the level of global sections provides a homomorphism

(3.3.1) $$\chi_\Delta := \chi_\Delta^\omega : E_\Delta \longrightarrow \mathrm{Hom}\big(E_{(\Delta,\partial\Delta)}, A[-2n]\big) ,$$

which is equivalent to (PD).

3.4 Theorem. [BBFK$_2$, 6.3] and [BreLu$_1$, 6.28] *Let the oriented fan Δ be normal, and fix a positive volume form $\omega \in \det V^*$. If a duality correlation $\varphi : \mathcal{E} \to \mathcal{D}\mathcal{E}$ is an isomorphism, then the induced pairing*

(PD) $$E_\Delta \times E_{(\Delta,\partial\Delta)} \longrightarrow A[-2n]$$

is a dual pairing of reflexive A-modules. If Δ is even quasi-convex, then the A-modules E_Δ and $E_{(\Delta,\partial\Delta)}$ are free, and thus the associated reduced pairing

($\overline{\text{PD}}$) $$\overline{E}_\Delta \times \overline{E}_{(\Delta,\partial\Delta)} \longrightarrow \overline{A}[-2n] \cong \mathbf{R}[-2n] \ .$$

is a dual pairing of graded real vector spaces.

Proof: Compose the isomorphisms φ_Δ and $\varphi_{(\Delta,\partial\Delta)}$ with the isomorphisms (2.3.1) and (2.3.2):

$$E_\Delta \xrightarrow{\cong} \mathcal{DE}_\Delta \cong \mathrm{Hom}(E_{(\Delta,\partial\Delta)}, A)$$

and

$$E_{(\Delta,\partial\Delta)} \xrightarrow{\cong} \mathcal{DE}_{(\Delta,\partial\Delta)} \cong \mathrm{Hom}(E_\Delta, A)\ .$$

If Δ is even quasi-convex, then the modules E_Δ and $E_{(\Delta,\partial\Delta)}$ are free, see [BBFK$_2$, 4.8, 4.12]. □

Theorem 3.2 now follows from this proposition with $\mathcal{F} = \mathcal{DE}$:

3.5 Proposition. [BBFK$_2$, 1.8 iii)] and [BreLu$_2$, 3.14] *For a fan Δ and two copies $\mathcal{E}$ and $\mathcal{F}$ of the equivariant intersection cohomology sheaf, every homomorphism*

$$\mathbf{R} = E_o \to F_o = \mathbf{R}$$

extends in a unique manner to a homomorphism $\mathcal{E} \to \mathcal{F}$ of degree 0.

For its proof, we need a Vanishing Lemma. This is the place where the Hard Lefschetz Theorem enters:

3.6 Lemma. [BBFK$_2$, 1.7, 1.8 ii)] and [BreLu$_2$, 3.13] *For the equivariant intersection cohomology sheaf $\mathcal{E}$ on a non-zero cone σ, the following equivalent conditions hold:*

(1) $\overline{E}^q_\sigma = 0$ *for* $q \geq \dim\sigma$,

(2) $\overline{E}^q_{(\sigma,\partial\sigma)} = 0$ *for* $q \leq \dim\sigma$,

(3) $E^q_{(\sigma,\partial\sigma)} = 0$ *for* $q \leq \dim\sigma$.

Proof: We may assume $\dim\sigma = n$.

(1) We use the setup of 0.3 (iii). First of all note that

$$B/\mathfrak{m}_B \otimes_B E_{\partial\sigma} \cong (B/\mathfrak{m}_B)[T] \otimes_{B[T]} E_{\partial\sigma}\ .$$

Now we tensorize the exact sequence

$$0 \longrightarrow (B/\mathfrak{m}_B)[T] \xrightarrow{\mu_T} (B/\mathfrak{m}_B)[T] \longrightarrow A/\mathfrak{m}_A \longrightarrow 0$$

with $E_{\partial\sigma}$ and obtain the exact sequence

$$(B/\mathfrak{m}_B) \otimes_B E_{\partial\sigma} \xrightarrow{\overline{\mu}_T} (B/\mathfrak{m}_B) \otimes_B E_{\partial\sigma} \longrightarrow A/\mathfrak{m}_A \otimes_A E_{\partial\sigma} \longrightarrow 0$$

with $\overline{\mu}_T := \mathrm{id}_{(B/\mathfrak{m}_B)} \otimes \mu_T$, where μ_T acts on the A-module $E_{\partial\sigma}$. Thus

$$\overline{E}_{\partial\sigma} \cong \mathrm{coker}\Big(\overline{\mu}_T : (B/\mathfrak{m}_B) \otimes_B E_{\partial\sigma} \longrightarrow (B/\mathfrak{m}_B) \otimes_B E_{\partial\sigma}\Big)\ .$$

On the other hand, according to [BBFK$_2$, (5.3.2)] together with (0.3.1) and using the notation of (1.3.1), we have an isomorphism

$$\overline{E}_{\partial\sigma} \;\cong\; \operatorname{coker}\Bigl(\overline{\mu}_\psi : IH(\Lambda_\sigma) \longrightarrow IH(\Lambda_\sigma)\Bigr) ,$$

where $\mu_\psi : \mathcal{E}(\Lambda_\sigma) \longrightarrow \mathcal{E}(\Lambda_\sigma)$ is the multiplication with the strictly convex conewise linear function

$$\psi := T \circ (\pi|_{\partial\sigma})^{-1} \;\in\; \mathcal{A}^2(\Lambda_\sigma) .$$

It now suffices to apply for $m := n{-}1$ the following theorem proved in [Ka]:

Hard Lefschetz Theorem. *Let Λ be a complete fan in the m-dimensional vector space V and $\psi \in \mathcal{A}^2(\Lambda)$ be a conewise linear strictly convex function. Then the homomorphism $L := \overline{\mu}_\psi$ induced by the multiplication $\mu_\psi : E_\Lambda \longrightarrow E_\Lambda$ with ψ induces isomorphisms*

$$L^k : IH^{m-k}(\Lambda) \longrightarrow IH^{m+k}(\Lambda)$$

for each $k \geq 0$. In particular L is injective in degrees $q \leq m-1$ and surjective in degrees $q \geq m-1$.

Let us finish the proof of 3.6: The equivalence of (1) and (2) follows from (2.6.1) and the dual pairing ($\overline{\mathrm{PD}}$) in Theorem 3.4 in the particular case $\Delta = \langle\sigma\rangle$, while the equivalence of (2) and (3) is a consequence of this fact: For a finitely generated graded A-module M, one has $M^q = 0$ for $q \leq r$ if and only if $\overline{M}^q = 0$ for $q \leq r$. ◻

Proof of Proposition 3.5: For an inductive proof, we have to show that over each non-zero cone σ, a homomorphism $\varphi_{\partial\sigma}\colon E_{\partial\sigma} \to F_{\partial\sigma}$ extends in a unique way to a homomorphism $\varphi_\sigma\colon E_\sigma \to F_\sigma$. By 3.6, (1), the A-modules E_σ and F_σ are generated by homogeneous elements of degree below $\dim\sigma$. On the other hand, 3.6, (3) yields $E^q_{(\sigma,\partial\sigma)} = 0 = F^q_{(\sigma,\partial\sigma)}$ for $q \leq \dim\sigma$. Hence, the restriction maps $E^q_\sigma \to E^q_{\partial\sigma}$ and $F^q_\sigma \to F^q_{\partial\sigma}$ are isomorphisms for $q < \dim\sigma$, whence the uniqueness of φ_σ follows. The existence is a consequence of the fact that E_σ is a free A_σ-module. ◻

4. Comparison with previous definitions

Let $\pi = \mathrm{id}_V : (V,\hat{\Delta}) \to (V,\Delta)$ be an *oriented refinement*, i.e., if a cone in $\hat{\Delta}^d$ is contained in a cone in Δ^d, then their orientations coincide.

4.1 Proposition. *For every pure sheaf $\mathcal{F}$ on $\hat{\Delta}$, there exists a canonical isomorphism*

$$\mathcal{D}\bigl(\pi_*(\mathcal{F})\bigr) \;\cong\; \pi_*(\mathcal{D}\mathcal{F}) .$$

Proof. For a cone $\sigma \in \Delta^d$, let $\hat{\sigma} \preceq \hat{\Delta}$ denote its refinement. Then formula (2.3.1), applied to $\hat{\sigma}$, yields the isomorphism in the following chain

$$\begin{aligned} \mathcal{D}\big(\pi_*(\mathcal{F})\big)_\sigma &= \operatorname{Hom}(\pi_*(\mathcal{F})_{(\sigma,\partial\sigma)}, A) \otimes \det V_\sigma^* = \operatorname{Hom}(F_{(\hat{\sigma},\partial\hat{\sigma})}, A) \otimes \det V_\sigma^* \\ &\cong \mathcal{D}\mathcal{F}_{\hat{\sigma}} = \pi_*(\mathcal{D}\mathcal{F})_\sigma \ . \end{aligned}$$

□

We now can prove the Compatibility Theorem:

4.2 Theorem. [BreLu$_2$, 7.2] *Let $\hat{\mathcal{E}}$ be the equivariant intersection cohomology sheaf of the oriented refinement $\hat{\Delta}$ of the normal n-dimensional fan Δ, and let $\iota : \mathcal{E} \to \pi_*(\hat{\mathcal{E}})$, a homomorphism of graded sheaves extending the identity $\mathcal{E}(o) = \mathbf{R} = \pi_*(\hat{\mathcal{E}})(o)$. Then the intersection products provide a commutative diagram*

$$\begin{array}{ccc} \mathcal{E}(\Delta) \times \mathcal{E}(\Delta, \partial\Delta) & \longrightarrow & \hat{\mathcal{E}}(\hat{\Delta}) \times \hat{\mathcal{E}}(\hat{\Delta}, \partial\hat{\Delta}) \\ & \searrow \quad \swarrow & \\ & A[-2n]\,. & \end{array}$$

Proof. The homomorphism ι provides a diagram

$$\begin{array}{ccc} \mathcal{E} & \stackrel{\iota}{\longrightarrow} & \pi_*(\hat{\mathcal{E}}) \\ \downarrow & & \downarrow \\ \mathcal{D}\mathcal{E} & \stackrel{\mathcal{D}\iota}{\longleftarrow} & \mathcal{D}\pi_*(\hat{\mathcal{E}}) \cong \pi_*(\mathcal{D}\hat{\mathcal{E}}) \end{array}$$

where the vertical arrows are ϑ respectively $\pi_*(\hat{\vartheta})$ with the duality correlations $\vartheta : \mathcal{E} \longrightarrow \mathcal{D}\mathcal{E}$ of (2.6.1) and $\hat{\vartheta} : \hat{\mathcal{E}} \longrightarrow \mathcal{D}\hat{\mathcal{E}}$. It is commutative at the zero cone and thus everywhere, see Proposition 3.5. Passing to the level of global sections yields the claim. □

Finally let us discuss the approach of [BBFK$_2$, 6.1]. Here we use the notion of an *evaluation map*:

4.3 Definition. *Let Δ be an oriented purely n-dimensional fan in the vector space V, endowed with a volume form $\omega \in \det V^*$. Then, for $1 \in E^0_\Delta = E^0_o = \mathbf{R}$, the homomorphism*

$$e^\omega_\Delta := \chi^\omega_\Delta(1) : E_{(\Delta,\partial\Delta)} \longrightarrow A[-2n] \ ,$$

see (3.3.1), *is called the* ***evaluation map*** *associated to ω.*

4.4 Theorem. *Let Δ be an oriented normal fan in a vector space V endowed with a volume form $\omega \in \det V^*$. Furthermore let*

$$\beta : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{E}$$

be a bilinear map of $\mathcal{A}$-module sheaves extending the multiplication

$$E_o \times E_o = \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R} = E_o$$

of real numbers. Then the pairing

$$e^{\omega}_{\Delta} \circ \beta_{\Delta} : \mathcal{E}(\Delta) \times \mathcal{E}(\Delta, \partial\Delta) \longrightarrow \mathcal{E}(\Delta, \partial\Delta) \longrightarrow A[-2n] \tag{4.4.1}$$

coincides with the intersection product.

Note that for a simplicial fan Δ, the equality $\mathcal{E} = \mathcal{A}$ holds, so the bilinear map β necessarily is the multiplication of functions and thus, symmetric. In the non-simplicial case, the map β is not uniquely determined. Nevertheless, there always exists such a map β that is symmetric. For a complete fan, the intersection product is thus symmetric, which also follows from Theorem 4.2 with a simplicial subdivision $\hat{\Delta}$ of Δ.

Proof. For each cone σ and a positive volume form $\omega_\sigma \in \det(V^*_\sigma)$, we define $e^{\omega_\sigma}_\sigma$ analogously to e^{ω}_{Δ}. Then $e^{\omega_\sigma}_\sigma \otimes \omega_\sigma$ does not depend on the choice of ω_σ, and the family of homomorphisms

$$\tilde{\varphi}_\sigma : \mathcal{E}_\sigma \longrightarrow \mathcal{D}\mathcal{E}_\sigma \ , \quad s \longmapsto \big(e^{\omega_\sigma}_\sigma \circ \beta\big)(s, \_) \otimes \omega_\sigma$$

defines a duality correlation $\tilde{\varphi} : \mathcal{E} \to \mathcal{D}\mathcal{E}$ and thus, according to Theorem 3.2, is unique. In particular the pairing (4.4.1) is the intersection product. ◻

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Addresses of authors

G. Barthel, L. Kaup
Fachbereich Mathematik
und Statistik
Universität Konstanz
Fach D 203
D-78457 Konstanz
e-mail:
`Gottfried.Barthel@uni-konstanz.de`
`Ludger.Kaup@uni-konstanz.de`

K.-H. Fieseler
Matematiska Institutionen
Box 480
Uppsala Universitet
SE-75106 Uppsala
e-mail: `khf@math.uu.se`

J.P. Brasselet.
IML/CNRS, Luminy Case 907
F-13288 Marseille Cedex 9
e-mail: `jpb@iml.univ-mrs.fr`